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KAC-MOODY GROUPS AND COMPLETIONS

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In memory of Kay Magaard

ABSTRACT. In this paper we construct a new “pro- p -complete” topological Kac-Moody group and compare it to various known topological Kac-Moody groups. We come across this group by investigating the process of completion of groups with BN-pairs. We would like to know whether the completion of such a group admits a BN-pair. We give explicit criteria for this to happen.

In this paper we study Kac-Moody groups over finite fields.

A Kac-Moody group is a generalisation of the notion of a reductive group to a more general Kac-Moody root datum \mathcal{D} or a closely related group with a BN-pair. Connected reductive groups are classified via a one-to-one correspondence to root data of finite type. A root datum of finite type yields a group scheme, generalised by Tits to a construction of a functor $G_{\mathcal{D}}$ from the category of commutative rings to the category of groups [T2, T3]. For instance, the Steinberg central extension $\mathrm{SL}_n(\widehat{\mathbb{F}_q[z, z^{-1}]})$ is a Kac-Moody group $G_{\mathcal{D}}(\mathbb{F}_q)$ for the simply-connected root datum \mathcal{D} of the affine type \tilde{A}_{n-1} (see [CKRu, sec. 6] for further details). On the other hand, $\mathrm{SL}_n(\mathbb{F}_q[z, z^{-1}])$ is not of the form $G_{\mathcal{D}}(\mathbb{F}_q)$, yet it is still called a Kac-Moody group.

A topological Kac-Moody group is a locally compact totally disconnected topological group that contains a Kac-Moody group. Often it is obtained by completion of a Kac-Moody group. In the examples of the previous paragraph, one arrives at topological Kac-Moody groups $\mathrm{SL}_n(\widehat{\mathbb{F}_q((z))})$ and $\mathrm{SL}_n(\mathbb{F}_q((z)))$.

There are several known topological Kac-Moody groups.¹ They are the Mathieu-Rousseau group G^{ma+} , the Carbone-Garland group $G^{c\lambda}$ and the Caprace-Rémy-Ronan group G^{crr} . Each of them contains $G_{\mathcal{D}}(\mathbb{F}_q)$. In this paper we show the existence of a new topological Kac-Moody group \widehat{G} .

Main Theorem. *Let A be an irreducible generalised Cartan matrix, \mathcal{D} a simply connected root datum of type A and $\mathbb{F} = \mathbb{F}_q$ a finite field of characteristic p ($q = p^a$, $a \in \mathbb{N}$). Let $G := G_{\mathcal{D}}(\mathbb{F}_q)$ be the corresponding minimal Kac-Moody group. Recall that it has a BN-pair (B, N) with $B = U \rtimes T$ where $U = \langle X_{\alpha} \mid \alpha \in \Delta_+^{re} \rangle$ (see Section 2, where the notations are introduced).*

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¹Since this paper a new book by Marquis [M3] was published, which would be a comprehensive source for further reading on the subject.

There exists a locally compact totally disconnected group \widehat{G} satisfying the following conditions:

- (1) G is a dense subgroup of \widehat{G} .
- (2) \widehat{G} has a BN-pair (\widehat{B}, N) where $\widehat{B} = \widehat{U} \rtimes T$ and \widehat{U} is the full pro- p completion of U .
- (3) If $\overline{G} = G^{crr}$ or $C^{c\lambda}$, or in the case when G is dense in G^{ma+} and $\overline{G} = G^{ma+}$, then there exists an open continuous surjective homomorphism $\widehat{G} \twoheadrightarrow \overline{G}$.
- (4) Let $Z'(\widehat{G}) := Z(G) \times C(\widehat{G})$ where $C(\widehat{G}) = \bigcap_{g \in \widehat{G}} \widehat{U}^g$.
 - (a) $\widehat{G}/Z'(\widehat{G})$ is a topologically simple group.
 - (b) If A is 2-spherical, then $\widehat{G}/Z'(\widehat{G})$ is an abstractly simple group.

Let us explain the content of the present paper. We investigate the process of completion of a group G with a BN-pair in Chapter 1. The main result is Theorem 1.1 that contains a sufficient condition for the completion \widehat{G} to inherit a BN-pair from G . It relies on Tits' description of groups with BN-pairs by generators and relations [T1]. The remainder of Chapter 1 contains several technical or user-friendly results about these completions. For instance, Theorem 1.2 conveniently constructs the completion \widehat{G} together with its BN-pair.

We put our completion results to good use in Chapter 2. After quickly recalling the definition of $G_{\mathcal{D}}(\mathbb{F}_q)$ we construct the new group $\widehat{G_{\mathcal{D}}(\mathbb{F}_q)}$. We compare it to other known completions and address its topological simplicity in Theorem 2.2 and its algebraic simplicity in Theorem 2.4. Thus, the Main Theorem is a combination of results in Section 2.

There have been previous attempts to compare various topological Kac-Moody groups: Capdeboscq and Rémy [CR], Baumgartner and Rémy [CarERi, 2.6], Marquis [M2], and Rousseau [Rou] all discuss the maps between different completions at length. We address these questions in Section 2 only modulo “congruence kernel” $Z'(\widehat{G})$ whose full computation remains mysterious. We devote the last chapter of the paper to several observations about $Z'(\widehat{G})$. Our major insight into the nature of the congruence kernel is its parabolic decomposition in Theorem 3.5.

Hristova and Rumynin study representations of topological Kac-Moody groups [HrRu]. The groups \widehat{G} are new examples for their theory.

1. COMPLETION THEOREM

Let (G, \mathcal{T}) be a Hausdorff topological group (G is a group, \mathcal{T} is a topology). The topology determines a right uniformity on G that we now describe following Bourbaki [B1]. Pick a basis \mathcal{J} of the topology at 1. The basis of uniformity \mathcal{T}^\diamond is $\mathcal{J}^\diamond = \{V^\diamond \mid V \in \mathcal{J}\}$ where $V^\diamond = \{(x, y) \in G^2 \mid xy^{-1} \in V\}$. The completion \widehat{G} is the set of all minimal Cauchy filters on $(G, \mathcal{T}^\diamond)$. Recall that a filter is a non-empty collection \mathcal{F} of open sets closed under intersections and oversets. A filter is Cauchy if it contains arbitrary “small” subsets, i.e., for each $V \in \mathcal{J}$ there exists $U \in \mathcal{F}$ such that $xy^{-1} \in V$ for all $x, y \in U$. We define the left uniformity ${}^\diamond\mathcal{T}$ in a similar way, using ${}^\diamond V = \{(x, y) \in G^2 \mid x^{-1}y \in V\}$ instead. The inverse map is an isomorphism of uniform spaces $\text{Inv} : (G, \mathcal{T}^\diamond) \rightarrow (G, {}^\diamond\mathcal{T})$ imposing an isomorphism between the right and left completions.

The completion \widehat{G} is always a monoid, although the multiplication $\text{Mult} : (G, \mathcal{T}^\diamond) \times (G, \mathcal{T}^\diamond) \rightarrow (G, \mathcal{T}^\diamond)$ is not uniformly continuous in general. It is a monoid because $\text{Mult}(\mathcal{F}, \mathcal{G})$ is a Cauchy filter for two Cauchy filters \mathcal{F}, \mathcal{G} on $(G, \mathcal{T}^\diamond)$ [B1, Prop. III.3.4(6)]. On the other hand, the completion is not necessarily a group: $\text{Inv} : (G, \mathcal{T}^\diamond) \rightarrow (G, \mathcal{T}^\diamond)$ is uniformly continuous but we have no information about uniform continuity of $\text{Inv} : (G, \mathcal{T}^\diamond) \rightarrow (G, \mathcal{T}^\diamond)$. The latter uniform continuity is a sufficient (but not necessary) condition for \widehat{G} to be a group. A necessary and sufficient condition is the following: if \mathcal{F} is a Cauchy filter on $(G, \mathcal{T}^\diamond)$, then $\text{Inv}(\mathcal{F})$ is a Cauchy filter on $(G, \mathcal{T}^\diamond)$ [B1, Th. III.3.4(1)].

For reader's convenience we sketch an example of a group G with non-a-group \widehat{G} following a hint in Bourbaki [B1, Exercise X.3(16)]. Let G be the group of auto-homeomorphisms of $[0, 1]$ with the topology of uniform convergence. It suffices to exhibit a uniformly convergent sequence f_m of homeomorphisms such that the sequence of inverses f_m^{-1} is not uniformly convergent. The following sequence fits the bill:

$$f_m(x) = \begin{cases} x^m & \text{if } x \leq \frac{1}{2} \\ (2 - 2^{1-m})x + (2^{1-m} - 1) & \text{if } x \geq \frac{1}{2} \end{cases}$$

Now suppose that G admits a BN-pair (B, N) . The key question is whether the completion \widehat{G} admits a BN-pair. Let \overline{B} be the closure of B in \widehat{G} . It is a moot point that \overline{B} is isomorphic to the completion of B in the restriction uniformity $\mathcal{T}^\diamond|_B$ [B1, Cor. II.3.9(1)]. A candidate BN-pair on \widehat{G} is (\overline{B}, N) but it does not work in general. Let \mathcal{G} be a simple split group scheme, $G = \mathcal{G}(\mathbb{F}[z, z^{-1}])$ its points over Laurent polynomials over a finite field, $N \leq G$ the group of monomial matrices, $I_- = [\mathcal{G}(\mathbb{F}[z^{-1}]) \xrightarrow{z^{-1} \mapsto 0} \mathcal{G}(\mathbb{F})]^{-1}(B)$ its negative Iwahori. The pair (I_-, N) is a BN-pair on G but $(\overline{I_-}, N) = (I_-, N)$ is not a BN-pair on the positive completion $\widehat{G} = \mathcal{G}(\mathbb{F}((z)))$: the countable groups I and N cannot generate uncountable \widehat{G} . A reader can see that the condition 3 of Theorem 1.1 fails for the negative Iwahori. On the other hand, if \mathbb{F} is finite, all conditions of Theorem 1.1 holds for the positive Iwahori $I_+ = [\mathcal{G}(\mathbb{F}[z]) \xrightarrow{z \mapsto 0} \mathcal{G}(\mathbb{F})]^{-1}(B)$ so that the theorem yields the standard BN-pair on \widehat{G} .

Nevertheless, we can prove the following partial affirmative answer, sufficiently general for the study of Kac-Moody groups. Let us explain some notations before stating the theorem. The notations (B, N) and $W = (W, S)$ are standard for the groups with BN-pairs. The homomorphism $\pi : N \rightarrow W$ is the natural surjection. For elements $s \in S$, $w \in W$ we choose some liftings $\dot{s} \in \pi^{-1}(s)$, $\dot{w} \in \pi^{-1}(w)$. The minimal parabolic P_s is the subgroup generated by B and \dot{s} .

Theorem 1.1. *Let G be a Hausdorff topological group with a BN-pair (B, N) with the Weyl group (W, S) where S is finite. If the following three conditions hold, then (\overline{B}, N) is a BN-pair on the completed group \widehat{G} .*

- (1) *The completion \widehat{G} is a group.*
- (2) *The index $|P_s : B|$ is finite for all $s \in S$.*
- (3) *B is open in G .*

Proof. We have systems of subgroups: $\mathfrak{A} = (B, N, P_s; s \in S)$ of G and $\mathfrak{B} = (\overline{B}, N, \overline{P_s}; s \in S)$ of \widehat{G} , where \overline{X} is the closure of X . The system of groups \mathfrak{A} satisfies all conditions of Tits' Theorem as observed by Tits [T1]. We claim that under the assumptions of this theorem the system \mathfrak{B} also satisfies these conditions.

We will verify this claim at the end of the proof. For the reader's convenience we restate Tits theorem (cf. [Ku, Th. 5.1.8]):

Tits Theorem. *Suppose that the system \mathfrak{B} satisfies the following conditions:*

- (P₁) *If $s \neq t \in S$, then $\overline{P}_s \cap \overline{P}_t = \overline{B}$.*
- (P₂) *The subgroup $\overline{B} \cap N$ is normal in N .*
- (P₃) *Given $s \in S$, let $N_s := \overline{P}_s \cap N$. Then $N_s/(\overline{B} \cap N)$ is of order 2 for all $s \in S$.*
- (P₄) *$\overline{P}_s = \overline{B} \cup \overline{B}s\overline{B}$ for all $s \in S$.*
- (P₅) *The pair $(N/(\overline{B} \cap N), S)$ is a Coxeter group.*
- (P₆) *Let $\pi : N \rightarrow W := N/(\overline{B} \cap N)$ be the quotient map. For any $n = \dot{s}_1 \dot{s}_2 \cdots \dot{s}_t \in N$ with $s_i \in S$ such that $s_1 s_2 \cdots s_t$ is a reduced word in W , the subgroup $\overline{B}(s_1, \dots, s_t)$ (see (1)) depends only on $w := \pi(n) = s_1 \cdots s_t \in W$ and the homomorphism $\overline{\gamma}(\dot{s}_1, \dots, \dot{s}_t)$ (see (2)) depends only on n . (This justifies the notation \overline{B}_w and $\overline{\gamma}_n$ from now on.)*
- (P₇) *If $w \in W$, $s \in S$ satisfy $l(ws) = l(w) + 1$, then $\overline{B}_w \overline{B}_s = \overline{B}_{ws}$.*
- (P₈) *If $w \in W$, $s, t \in S$ satisfy $wtw^{-1} = s$ and $l(wt) = l(w) + 1$, then for all $x \in \pi^{-1}(s)$, $n \in \pi^{-1}(w)$ and $b \in \overline{B} \setminus \overline{B}_t$, there exist $y \in b\overline{B}_t \cap \overline{B}_n$ and $y', y^\sharp \in \overline{B}_n$ such that*
 - (a) $x'^{-1}yx' = y'x'y^\sharp$ *in \overline{P}_t and*
 - (b) $x\overline{\gamma}_n(y)x^{-1} = \overline{\gamma}_n(y')x^{-1}\overline{\gamma}_n(y^\sharp)$ *in \overline{P}_s .**where $x' := n^{-1}x^{-1}n \in \pi^{-1}(t)$.*
- (P₉) *\overline{B} is not normal in any \overline{P}_s .*

Then the canonical map

$$N \cup (\cup_{s \in S} \overline{P}_s) \longrightarrow \tilde{G} := {}_{\mathfrak{B}}^* H$$

to the amalgam \tilde{G} is injective. The amalgam \tilde{G} admits a BN-pair (\overline{B}, N) with a set of simple reflections S , where we identify the groups \overline{B} and N with their images in \tilde{G} under the canonical map.

Furthermore, consider a group G' and an injective function

$$\varphi : N \cup (\cup_{s \in S} \overline{P}_s) \longrightarrow G'$$

such that $\varphi|_N$ and all $\varphi|_{\overline{P}_s}$ are group homomorphisms. If G' is generated by the image of φ , then the canonical homomorphism $\varphi^\sharp : \tilde{G} \rightarrow G'$ is an isomorphism.

Tits' Theorem applies to $G' = \hat{G}$, allowing us to conclude that the group \tilde{G} admits a BN-pair (\overline{B}, N) and the natural group homomorphism $f : \tilde{G} \rightarrow \hat{G}$ is injective.

It remains to check surjectivity of f . The image $f(\tilde{G})$ contains N and B , which generate G . Hence, $f(\tilde{G})$ is dense in \hat{G} . On the other hand, $f(\tilde{G})$ contains \overline{B} , which is open in \hat{G} because B is open in G . Thus, $f(\tilde{G})$ is open in \hat{G} but it is a subgroup, hence, $f(\tilde{G})$ is also closed in \hat{G} . Being closed and dense, $f(\tilde{G})$ must be equal to \hat{G} .

It only remains to verify all nine conditions in Tits' Theorem for \mathfrak{B} . Our starting point is that these nine conditions hold in G for \mathfrak{A} as shown by Tits [T1].

(P₂) We know that $B \cap N$ is normal in N . Clearly, $B \cap N \subseteq \overline{B} \cap N$. In the opposite direction, $\overline{B} \cap N = (\overline{B} \cap G) \cap N$. An element $x \in \overline{B} \cap G$ is a limit of a net

$$x = \lim_{m \in \mathbb{M}} b_m, \quad b_m \in B, \quad \mathbb{M} \text{ is an ordinal.}$$

This limit works in G as well where B is open, hence, closed. Thus, $x \in B$ and $\overline{B} \cap G = B$. Therefore, $\overline{B} \cap N = B \cap N$ is normal in N .

(P₁) Let $s \neq t \in S$. Since $P_s = B\dot{s}B \cup B$, we conclude that $\overline{P_s} = \overline{B\dot{s}B} \cup \overline{B}$. Let us prove now that $\overline{B\dot{s}B} \cap \overline{B\dot{t}B} = \emptyset$. An element $x \in \overline{B\dot{s}B} \cap \overline{B\dot{t}B}$ is a limit of two nets

$$x = \lim_{m \in \mathbb{M}} a_m \dot{s} b_m = \lim_{m \in \mathbb{M}} c_m \dot{t} d_m, \quad a_m, b_m, c_m, d_m \in B.$$

Since \overline{B} is open there exists an ordinal $\mathbb{L} < \mathbb{M}$ such that $a_m \dot{s} b_m x^{-1} \in \overline{B} \ni x(c_m \dot{t} d_m)^{-1}$ and consequently $a_m \dot{s} b_m d_m^{-1} \dot{t}^{-1} c_m^{-1} \in \overline{B}$ for all $m \geq \mathbb{L}$. Clearly $a_m \dot{s} b_m d_m^{-1} \dot{t}^{-1} c_m^{-1} \in G$. It is shown (P₂) that $\overline{B} \cap G = B$, thus, $a_m \dot{s} b_m d_m^{-1} \dot{t}^{-1} c_m^{-1} \in B$ for all $m \geq \mathbb{L}$. On the other hand, these elements $a_m \dot{s} b_m d_m^{-1} \dot{t}^{-1} c_m^{-1}$ lie in $B\dot{s}B\dot{t}B$ equal to the double coset $B\dot{s}\dot{t}B$ since $l(\dot{s}\dot{t}) = 2 = l(\dot{s}) + l(\dot{t})$. Since $B\dot{s}B\dot{t}B \cap B = \emptyset$, no such x exists. Therefore, $\overline{P_s} \cap \overline{P_t} = \overline{B}$.

(P₃) The minimal parabolic P_s is a union of cosets of B , hence, open in G . Similarly to the proof in (P₂), $N_s = \overline{P_s} \cap N$ is equal to $(\overline{P_s} \cap G) \cap N = P_s \cap N$. Therefore, $N_s/(\overline{B} \cap N)$ is of order 2 for all $s \in S$.

(P₄) By condition (2), B has finite index in P_s . Hence $B\dot{s}B = XB$ for some finite subset $X \subseteq P_s$. Observe now that

$$\overline{B\dot{s}B} \subseteq \overline{B\dot{s}B} = \overline{XB} \stackrel{f}{\supseteq} X\overline{B} \subseteq \overline{B\dot{s}B}.$$

Since X is finite, $X\overline{B}$ is closed and the inclusion f is an equality. Thus, $\overline{B\dot{s}B} = \overline{B\dot{s}B}$. Therefore, $\overline{P_s} = \overline{B} \cup \overline{B\dot{s}B} = \overline{B} \cup \overline{B\dot{s}B}$.

(P₅) We have proved in (P₂) that $\overline{B} \cap N = B \cap N$. Therefore, $(N/\overline{B} \cap N, S) = (W, S)$ is a Coxeter group.

(P₆) Let us first observe that if $A, K \leq G$ are open subgroups, then $\overline{A} \cap \overline{K} = \overline{A \cap K}$ in \widehat{G} . Indeed, the inclusion \supseteq is obvious. To prove the inclusion \subseteq consider $x \in \overline{A} \cap \overline{K}$. It is a limit of two nets

$$x = \lim_{m \in \mathbb{M}} a_m = \lim_{m \in \mathbb{M}} k_m, \quad a_m \in A, k_m \in K.$$

Then the net $a_m k_m^{-1} = (a_m x^{-1})(x k_m^{-1})$ converges to $1 \in G$. Since $A \cap K$ is open there exists an ordinal $\mathbb{L} < \mathbb{M}$ such that $a_m k_m^{-1} \in A \cap K$, and consequently $a_m, k_m \in A \cap K$ for all $m \geq \mathbb{L}$. Thus, $x \in \overline{A \cap K}$.

The following subgroups are defined recursively for a reduced word $s_1 s_2 \cdots s_t \in W$ and its fixed lift $n = \dot{s}_1 \dot{s}_2 \cdots \dot{s}_t \in N$

$$B(s_1, \dots, s_i) := B \cap \dot{s}_i^{-1} B(s_1, \dots, s_{i-1}) \dot{s}_i,$$

$$(1) \quad \overline{B}(s_1, \dots, s_i) := \overline{B} \cap \dot{s}_i^{-1} \overline{B}(s_1, \dots, s_{i-1}) \dot{s}_i.$$

The aforementioned observation implies that $\overline{B}(s_1, \dots, s_t) = \overline{B(s_1, \dots, s_t)}$. Consequently, the homomorphism

$$(2) \quad \overline{\gamma}(\dot{s}_1, \dots, \dot{s}_t) : \overline{B}(s_1, \dots, s_t) \rightarrow \overline{B}, \quad x \mapsto \dot{s}_1 \cdots \dot{s}_t x \dot{s}_t^{-1} \cdots \dot{s}_1^{-1}$$

is uniquely determined by its restriction $\gamma(\dot{s}_1, \dots, \dot{s}_t) : B(s_1, \dots, s_t) \rightarrow B$. Property (P₆) hold for \mathfrak{A} . This means that $B(s_1, \dots, s_t)$ depends only on the element $w = s_1 \cdots s_t \in W$, not the word or the choice of the liftings $\dot{s}_i \in N$. It also means that $\gamma(\dot{s}_1, \dots, \dot{s}_t)$ depends only on the element $n = \dot{s}_1 \cdots \dot{s}_t \in N$. Therefore, the subgroup $\overline{B}(s_1, \dots, s_t)$ and the homomorphism $\overline{\gamma}(\dot{s}_1, \dots, \dot{s}_t)$ depend only on $w \in W$ and $n \in N$ correspondingly. (We denote these $B_w, \overline{B}_w, \gamma_n, \overline{\gamma}_n$.)

(P₇) We begin by proving that all subgroups B_w , $w \in W$ are commensurable. We proceed by induction on the length $l(w)$ to show that B_w has finite index in B .

If $l(w) = 1$, then $w = s$ for some $s \in S$. Since $xB_s \mapsto x\dot{s}B$ is an embedding of quotient sets $B/B_s \hookrightarrow P_s/B$, we conclude that $|B : B_s| \leq |P_s : B| < \infty$ by assumption (2).

Suppose the case of $l(w) = m - 1$ is settled. Consider $w \in W$, $s \in S$ with $l(ws) = m$. Then $B_{ws} = B \cap \dot{s}^{-1}B_w\dot{s}$. Hence,

$$|B : B_{ws}| = |B : B_s| |(B \cap \dot{s}^{-1}B_w\dot{s}) : (B \cap \dot{s}^{-1}B_w\dot{s})| \leq |B : B_s| |B : B_w| < \infty$$

since by induction assumption $|B : B_w| < \infty$.

Property (P₇) for \mathfrak{A} ensures that $B_w B_s = B$ if $l(ws) = l(w) + 1$ as before. Hence,

$$\overline{B} = \overline{B_w B_s} \supseteq \overline{B_w} \overline{B_s} = \overline{B_w} \overline{B_s}$$

because $\overline{XY} \supseteq \overline{X} \overline{Y}$ for all subsets X, Y and $\overline{B_w} = \overline{B_w}$ for all $w \in W$ as shown in (P₆). Since B_w and B_s are commensurable, $B_w B_s = XB_s$ for a finite subset $X \subseteq B_w$. Then

$$\overline{B_w B_s} = \overline{XB_s} = X \overline{B_s} \subseteq \overline{B_w} \overline{B_s}$$

since $X \overline{B_s}$ is closed as a finite union of closed cosets of $\overline{B_s}$. Therefore, $\overline{B_w} \overline{B_s} = \overline{B}$.

(P₈) Let $s, t \in S$, $w \in W$ such that $wtw^{-1} = s$ and $l(wt) = l(w) + 1$. Let us fix arbitrary $x \in \pi^{-1}(s)$, $n \in \pi^{-1}(w)$ and define $x' := n^{-1}x^{-1}n \in \pi^{-1}(t)$. Now pick any $b \in \overline{B} \setminus \overline{B_t}$. As shown in (P₇), $B = XB_t$ for a finite set X . Without loss of generality, $1 \in X$ and $B \setminus B_t = (X \setminus \{1\})B_t$. By the argument as above ($X\overline{B_t}$ is closed etc.), $\overline{B} \setminus \overline{B_t} = (X \setminus \{1\})\overline{B_t}$. Hence, $b = b_1 b_2$ for some $b_1 \in X \setminus \{1\}$ and $b_2 \in \overline{B_t}$. This brings property (P₈) down to the system \mathfrak{A} where we know it [T1]. Therefore, there exist elements $y \in b_1 B_t \cap B_w \subseteq b \overline{B_t} \cap \overline{B_w}$ and $y', y^\# \in B_w \subseteq \overline{B_w}$ satisfying $x'^{-1}yx' = y'x'y^\# \in P_t \subseteq \overline{P_t}$ and $x\gamma_n(y)x^{-1} = \gamma_n(y')x^{-1}\gamma_n(y^\#) \in P_s \subseteq \overline{P_s}$.

(P₉) Recall that $\overline{B} \cap G = B$ and $\overline{P_s} \cap G = P_s$ as shown in (P₂). If \overline{B} were normal in $\overline{P_s}$, then B would be normal in P_s , contradicting (P₉) for \mathfrak{A} . Therefore, \overline{B} is not normal in $\overline{P_s}$. \square

A shortcoming of Theorem 1.1 is that it requires the group \widehat{G} to exist first. It would be useful to tweak the theorem to enable construction of new groups. The next theorem addresses this issue. If \mathcal{T} is a topology on a group B , we denote $\mathcal{T}_1 := \{A \in \mathcal{T} \mid 1 \in A\}$.

Theorem 1.2. *Let G be a group with a BN-pair (B, N) with the Weyl group (W, S) where S is finite. Suppose further that a topology \mathcal{T} on B is given such that the four conditions (1)–(4) hold.*

- (1) (B, \mathcal{T}) is a topological group.
- (2) The completion \widehat{B} is a group.
- (3) \mathcal{T}_1 is a basis at 1 of topology on each minimal parabolic P_s , $s \in S$ that defines a structure of topological group on P_s .
- (4) The index $|P_s : B|$ is finite for each $s \in S$.

Under these conditions the following statements hold:

- (a) \mathcal{T}_1 is a basis at 1 of topology on G that defines a structure of topological group on G .
- (b) The completion \widehat{G} is a group and $\widehat{B} = \overline{B}^{\subseteq \widehat{G}}$.

(c) The completion \widehat{G} is isomorphic to the amalgam $*_{\mathfrak{B}} H$ where

$$\mathfrak{B} = \{\overline{B}, N, \overline{P_s}; s \in S\}.$$

(d) The pair (\overline{B}, N) is a BN-pair on the completed group \widehat{G} .

Proof. (a) We already know that \mathcal{T}_1 is a filter of neighbourhoods of 1 in a topological group B . To verify (a) it suffices to show that for all $g \in G$, $A \in \mathcal{T}_1$ it holds that $gAg^{-1} \in \mathcal{T}_1$ [B1, Prop. III.1.2(1)]. By (3) we know this property for all $g \in P_s$. Since G is generated by all P_s , we conclude the proof.

(b) Denote the aforementioned topology on G by \mathcal{T}_G . We need to show that the monoid $(\widehat{G}, \widehat{\mathcal{T}_G})$ is a group. Consider $x \in (\widehat{G}, \widehat{\mathcal{T}_G})$ and a convergent net $x_m \rightarrow x$, $m \in \mathbb{M}$, $x_m \in G$. Since $B \in \mathcal{T}_1$, B is open in (G, \mathcal{T}_G) . The net x_m is Cauchy, so there exists an ordinal \mathbb{L} such that $x_m x_{\mathbb{L}}^{-1} \in B$ for all $m \geq \mathbb{L}$. Let

$$y_m = \begin{cases} 1 & \text{if } m < \mathbb{L}, \\ x_m x_{\mathbb{L}}^{-1} & \text{if } m \geq \mathbb{L}. \end{cases}$$

Since $y_m y_l^{-1} = (x_m x_{\mathbb{L}}^{-1})(x_l x_{\mathbb{L}}^{-1})^{-1} = x_m x_l^{-1}$, the net y_m is a Cauchy net in B . Let $y = \lim y_m \in \widehat{B}$. The inverse $y^{-1} \in \widehat{B}$ exists because \widehat{B} is a group. Then

$$(x_{\mathbb{L}}^{-1} y^{-1}) \cdot x = x_{\mathbb{L}}^{-1} \cdot \lim_m y_m^{-1} \cdot \lim_m x_m = x_{\mathbb{L}}^{-1} \cdot \lim_m y_m^{-1} x_m = x_{\mathbb{L}}^{-1} \cdot \lim_m x_{\mathbb{L}} = 1.$$

Similarly, $x \cdot (x_{\mathbb{L}}^{-1} y^{-1})$. Thus, we have found the inverse $x^{-1} = x_{\mathbb{L}}^{-1} y^{-1} \in \widehat{G}$ so that \widehat{B} is a group. Coincidence of the completion and the closure is standard.

(c+d) These follow immediately from Theorem 1.1. \square

Suppose that a group G admits two topological group structures (G, \mathcal{S}) and (G, \mathcal{T}) such that $\mathcal{S} \subseteq \mathcal{T}$. Then the identity map $\text{Id} : (G, \mathcal{T}) \rightarrow (G, \mathcal{S})$ is a homomorphism of topological groups that admits a unique extension $\widehat{\text{Id}} : (\widehat{G}, \widehat{\mathcal{T}}) \rightarrow (\widehat{G}, \widehat{\mathcal{S}})$ [B1, Prop. III.3.4(8)]. This extension $\widehat{\text{Id}}$ may or may not be injective in general. Ditto for surjective [B1, Exercise III.3(12)]. However, we can give nice criteria for surjectivity and injectivity for the topologies we are interested in.

Corollary 1.3. *Consider a group G with a BN-pair (B, N) that admits two topological group structures (G, \mathcal{S}) and (G, \mathcal{T}) such that $\mathcal{S} \subseteq \mathcal{T}$. Suppose $B \in \mathcal{S}$. Then the kernel of $\widehat{\text{Id}} : (\widehat{G}, \widehat{\mathcal{T}}) \rightarrow (\widehat{G}, \widehat{\mathcal{S}})$ is equal to the kernel of $\widehat{\text{Id}}_B : (\widehat{B}, \widehat{\mathcal{T}}_B) \rightarrow (\widehat{B}, \widehat{\mathcal{S}}_B)$.*

Proof. Clearly, $\ker(\widehat{\text{Id}}) \supseteq \ker(\widehat{\text{Id}}_B)$. In the opposite direction, consider $x \in \ker(\widehat{\text{Id}})$. This element is a limit of a Cauchy net $x_m \in G$, $m \in \mathbb{M}$ in \mathcal{T} such that $x_m \rightarrow 1$ in \mathcal{S} . Since $B \in \mathcal{S}$ there exists an ordinal $\mathbb{L} < \mathbb{M}$ such that $x_m \in B$ for all $m \geq \mathbb{L}$. Thus, $x \in (\widehat{B}, \widehat{\mathcal{T}}_B)$ and $x \in \ker(\widehat{\text{Id}}_B)$. \square

Corollary 1.4. *Consider a group G with a BN-pair (B, N) that admits two topological group structures (G, \mathcal{S}) and (G, \mathcal{T}) such that $\mathcal{S} \subseteq \mathcal{T}$. Suppose (G, \mathcal{S}) satisfies the conditions of Theorem 1.1 or Theorem 1.2. If $\widehat{\text{Id}}_B : (\widehat{B}, \widehat{\mathcal{T}}_B) \rightarrow (\widehat{B}, \widehat{\mathcal{S}}_B)$ is surjective, then $\widehat{\text{Id}} : (\widehat{G}, \widehat{\mathcal{T}}) \rightarrow (\widehat{G}, \widehat{\mathcal{S}})$ is surjective.*

Proof. This holds because $(\widehat{G}, \widehat{\mathcal{S}})$ is generated by N and \overline{B} . \square

Surjectivity of $\widehat{\text{Id}}$ has a very interesting consequence as pointed out to us by Guy Rousseau. Note that the map $\widehat{\text{Id}} : (\widehat{G}, \widehat{\mathcal{T}}) \rightarrow (\widehat{G}, \widehat{\mathcal{S}})$ defines an injective map of Tits

buildings $\mathcal{TB}(\widehat{(G, \mathcal{T})}) \rightarrow \mathcal{TB}(\widehat{(G, \mathcal{S})})$. Surjectivity of $\widehat{\text{Id}}$ implies that this map of Tits buildings is bijective.

We finish this section with a convenient corollary of Theorem 1.2 whose proof is straightforward.

Corollary 1.5. *Let G be a group with a BN -pair (B, N) with the Weyl group (W, S) where S is finite. Suppose further that a system \mathfrak{S} of subgroups of B is given such that the following three conditions hold.*

- (1) \mathfrak{S} forms a topology basis at 1 of a topological group (B, \mathcal{T}) .
- (2) Each minimal parabolic P_s is split as a semidirect product $P_s = L_s \rtimes U_s$ where L_s is a finite group and U_s is a subgroup of B .
- (3) L_s acts continuously on (U_s, \mathcal{T}_{U_s}) .

Then the four conclusions of Theorem 1.2 hold.

2. COMPLETIONS OF KAC-MOODY GROUPS

Let $A = (a_{ij})_{n \times n}$ be a generalised Cartan matrix, $\mathcal{D} = (I, A, \mathcal{X}, \mathcal{Y}, \Pi, \Pi^\vee)$ a root datum of type A . Recall that this means

- $I = \{1, 2, \dots, n\}$,
- \mathcal{Y} is a free finitely generated abelian group,
- $\mathcal{X} = \mathcal{Y}^* = \text{hom}(\mathcal{Y}, \mathbb{Z})$ is its dual group,
- $\Pi = \{\alpha_1, \dots, \alpha_n\}$ is a set of simple roots, where $\alpha_i \in \mathcal{X}$,
- $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ is a set of simple coroots, where $\alpha_i^\vee \in \mathcal{Y}$,
- for all $i, j \in I$, $\alpha_j(\alpha_i^\vee) = a_{ij}$.

Recall that \mathcal{D} is *simply connected* if Π^\vee is a basis of \mathcal{Y} . We call A 2-spherical if for each $J \subseteq I$ with $|J| = 2$, the submatrix $A_J := (a_{ij})_{i, j \in J}$ is a Cartan matrix of finite type. Let Δ^{re} be the set of real roots. Recall that $\Delta^{re} = W(\Pi)$ where W is the Weyl group. Note that $\Delta^{re} = \Delta_+^{re} \cup \Delta_-^{re}$.

Let $\mathbb{F} = \mathbb{F}_q$ be a finite field of $q = p^a$ elements ($a \geq 1$ and p a prime). Tits [T2, T3] gives a definition of a Kac-Moody group $G := G_{\mathcal{D}}(\mathbb{F})$, which is generated by the torus $T = \text{hom}(\mathcal{X}, \mathbb{F}^\times)$ and root subgroups $X_\alpha \cong \mathbb{F}^+$, $\alpha \in \Delta^{re}$. For all $i \in I$, set

$$M_i := \langle X_{\alpha_i} \cup X_{-\alpha_i} \rangle.$$

The group G admits a BN -pair (B, N) where $B = U \rtimes T$ with $U := \langle X_\alpha \mid \alpha \in \Delta_+^{re} \rangle$ and $N := N_G(T)$, the normaliser of T . If \mathcal{D} is simply connected, $T \cong (\mathbb{F}^\times)^n$. Moreover, the Coxeter group $(N/T, S = \{s_i, i \in I\})$ and (W, S) are isomorphic. We denote by $P_i := P_{s_i}$, $i \in I$, a minimal parabolic subgroup of G . It is known [CaR2, 6.2] that $P_i = U_i \rtimes L_i$ where $L_i := M_i T$ and $U_i = U \cap s_i U s_i^{-1}$. In particular, $|P_i : B|$ is finite for all $i \in I$.

Consider the set of subgroups \mathcal{F} of B where

$$\mathcal{F} = \{A \mid A \trianglelefteq U \leq B, |U : A| = p^a \text{ for some } a \in \mathbb{N}\}.$$

Elements of \mathcal{F} form a basis at 1 of a topology on B . Then the completion \widehat{B} of B with respect to this topology is a group. Since $|P_s : U| < \infty$, conditions (1)–(4) of Theorem 1.2 are satisfied, thus its conclusions hold. In particular, \widehat{G} is a topological group with an open subgroup \widehat{B} and (\widehat{B}, N) is a BN -pair of \widehat{G} .

Since U is a residually finite- p group [E, Remark after Th. 4.1] and $|B : U| \mid (q - 1)^n$, our completion \widehat{B} is equal to $\widehat{U} \rtimes T$ where \widehat{U} is the full pro- p completion of U .

Let us recall other known topological Kac-Moody groups. They are the Mathieu-Rousseau group G^{ma+} , the Carbone-Garland group $G^{c\lambda}$ and the Caprace-R  my-Ronan group G^{crr} . Each of them contains a quotient $G^\dagger := G/Z$ by a central subgroup Z , which depends on the completion and could be trivial. In fact, G^\dagger is always dense in $G^{c\lambda}$ and G^{crr} . Let \overline{G}^+ be the closure of G^\dagger in G^{ma+} . Rousseau [Rou, 6.10] investigates whether \overline{G}^+ equals G^{ma+} and show that this happens when $p > \max\{|a_{ij}|, i \neq j\}$ [Rou, 6.11]. Rousseau and later Marquis give examples when it does not happen [Rou], [M2].

There are two further known completions of G where the closure \overline{U} is compact totally disconnected [ReW]. The Belyaev group G^b is the “largest” such completion. The Schlichting group G^s is the “smallest” such completion. Our completion admits a characterisation similar to the Belyaev group: \widehat{G} is the “largest” completion where the closure \overline{U} is a pro- p -group.

Let \overline{U} and \overline{B} be the closures of U and B correspondingly in either of the topological groups \overline{G}^+ , $G^{c\lambda}$ or G^{crr} . The group homomorphism $\widehat{U} \rightarrow \overline{U}$ extends to $\widehat{B} \rightarrow \overline{B}$ and $\widehat{G} \rightarrow \overline{G}$ (cf. Section 6.3 of [Rou]). Using this and the universal properties of the Belyaev and Schlichting completions, we have open continuous homomorphisms [Rou, 6.3]:

$$(3) \quad G^b \twoheadrightarrow \widehat{G} \twoheadrightarrow \overline{G}^+ \twoheadrightarrow G^{c\lambda} \twoheadrightarrow G^{crr} \xrightarrow{\cong} G^s.$$

It is known that for $\overline{G} \in \{\overline{G}^+, G^{c\lambda}, G^{crr}\}$, $\overline{G}/Z'(\overline{G})$ is topologically simple (where $Z'(\overline{G}) = \bigcap_{g \in \overline{G}} g\overline{B}g^{-1}$). What about our new group \widehat{G} ?

Recall the following criterion of Bourbaki [B2].

Proposition 2.1. *Let (G, B, N, S) be a Tits system with Weyl group $W = N/(B \cap N)$. Let U be a subgroup of B . We set $Z'(G) = \bigcap_{g \in G} gBg^{-1}$. Assume that G is a topological group topologically generated by the conjugates of U in G . Assume further B a closed subgroup of G , and the following conditions hold:*

- (1) *We have $U \triangleleft B$ and $B = UT$ where $T = B \cap N$.*
- (2) *For any proper normal closed subgroup $V \triangleleft U$, we have $[U/V, U/V] \subsetneq U/V$.*
- (3) *Subgroup $[G, G]$ is dense in G .*
- (4) *The Coxeter system (W, S) is irreducible.*

Then for any normal closed subgroup K in G , $K \leq Z'(G)$. In particular, $G/Z'(G)$ is topologically simple.

We now prove the following statement.

Theorem 2.2. *If A is an irreducible generalised Cartan matrix, then $\widehat{G}/Z'(\widehat{G})$ is topologically simple.*

Proof. If V is a closed normal subgroup of \widehat{U} , then $[\widehat{U}/V, \widehat{U}/V] \neq \widehat{U}/V$ as shown in [CarERi, 4.4]. Now Proposition 2.1 finishes the proof. \square

There is a similar criterion for the abstract simplicity [B2].

Proposition 2.3. *Let (G, B, N, S) be a Tits system with Weyl group $W = N/(B \cap N)$. Let U be a subgroup of B such that G is generated by the conjugates of U . Assume that the following holds.*

- (1) *We have $U \triangleleft B$ and $B = UT$ where $T = B \cap N$.*
- (2) *For any proper normal subgroup $V \triangleleft U$, we have $[U/V, U/V] \subsetneq U/V$.*

(3) We have $G = [G, G]$.

(4) The Coxeter system (W, S) is irreducible.

Then for any normal subgroup K in G , $K \leq Z'(G)$. In particular, $G/Z'(G)$ is abstractly simple.

This allows us to prove the following statement.

Theorem 2.4. *Suppose $q \geq 4$. If A is irreducible and 2-spherical, then $\widehat{G}/Z'(\widehat{G})$ is abstractly simple, and there are natural isomorphisms*

$$\widehat{G}/Z'(\widehat{G}) \xrightarrow{\cong} \overline{G}^+/Z'(\overline{G}^+) \xrightarrow{\cong} G^{c\lambda}/Z'(G^{c\lambda}) \xrightarrow{\cong} G^{crr}/Z'(G^{crr}).$$

Proof. Let us first show that $\widehat{G}/Z'(\widehat{G})$ is abstractly simple. To do that it suffices to check the conditions of Proposition 2.3 for the Tits system $(\widehat{G}, \widehat{B}, N, S)$.

By construction of \widehat{G} , condition (1) holds because it holds in (G, B, N, S) .

Abramenko proves that for $q \geq 4$, U is finitely generated if and only if A is 2-spherical [A]. Thus, \widehat{U} is topologically finitely generated. By [CarERi, Lemma 4.4], condition (2) of Proposition 2.3 holds for \widehat{U} and any proper normal subgroup V of \widehat{U} .

Moreover, $[\widehat{G}, \widehat{G}] \geq [G, G][\widehat{U}, \widehat{U}]$. Since $q \geq 4$, for every $\alpha \in \Delta^{re}$, the subgroup $M_\alpha := \langle X_\alpha, X_{-\alpha} \rangle$ is perfect (in fact, it is $\mathrm{PSL}_2(\mathbb{F})$ or $\mathrm{SL}_2(\mathbb{F})$), and thus $[M_\alpha, M_\alpha] = M_\alpha \supseteq X_\alpha$. Hence, $G = [G, G]$. Now the argument of Carbone, Ershov and Ritter [CarERi, 4.3(b)] shows that $[\widehat{U}, \widehat{U}]$ is an open subgroup of \widehat{G} , and so $\widehat{G} = [\widehat{G}, \widehat{G}]$.

Finally, condition (4) holds since A is irreducible. Therefore, $\widehat{G}/Z'(\widehat{G})$ is abstractly simple.

Observe that the homomorphisms (3) yield open surjective homomorphisms

$$\widehat{G}/Z'(\widehat{G}) \twoheadrightarrow \overline{G}^+/Z'(\overline{G}^+) \twoheadrightarrow G^{c\lambda}/Z'(G^{c\lambda}) \twoheadrightarrow G^{crr}/Z'(G^{crr})$$

that are isomorphisms of abstract groups due to simplicity of $\widehat{G}/Z'(\widehat{G})$. They are isomorphisms of topological groups because they are open. \square

3. CONGRUENCE KERNEL

We finish the paper with some observations on the structure of $Z'(\widehat{G})$. To facilitate our discussion we use the following notation for arbitrary groups $K \leq H$:

- \widehat{H} – the completion of H in the pro- p topology on H or its canonical (such as U) subgroup,
- \widetilde{H} – the completion of H in some other topology,
- $\overline{K}^{\subseteq H}$ (or simply \overline{K}) – the closure of K in H ,
- $C(H, K) := \cap_{g \in H} gKg^{-1}$ – the normal core of K in H .

The group $Z'(\widehat{G})$ contains two commuting subgroups: the centre (before completion) $Z(G)$ and the normal core $C(\widehat{G}, \widehat{U})$. In fact, as $\widehat{U} \cong \overline{U}^{\subseteq \widehat{G}}$ is a Sylow pro- p subgroup of \widehat{G} , $C(\widehat{G}, \widehat{U}) = C(\widehat{G}, V)$ for any Sylow pro- p subgroup V of \widehat{G} . Therefore, we may use the notation $C(\widehat{G})$ instead of $C(\widehat{G}, \widehat{U})$. Sometimes it is convenient to use the full notation $C(\widehat{G}, \widehat{U})$. We will use both notations depending on circumstances.

Following the argument of Rousseau [Rou, Prop. 6.4], we can prove that

$$Z'(\widehat{G}) = Z(G) \times C(\widehat{G}).$$

We can compute the centre $Z(G)$ from the Cartan matrix but we see no efficient way of computing the normal core $C(\widehat{G})$. Observe that in the Caprace-R  my-Ronan completion, $Z'(G^{crr}) = Z(G)$. Hence, Theorem 2.4 implies that $C(\widehat{G}) = \ker(\phi)$ where $\phi : \widehat{G} \rightarrow G^{crr}$ is the natural continuous open surjective homomorphism. The kernels of the natural maps between two different completions of the same groups are commonly known as *congruence kernels*, the term used later in the paper. Can we describe $C(\widehat{G})$ explicitly?

Let \mathcal{P} be the collection of all normal index p^n , $n \in \mathbb{N}$, subgroups of U so that

$$\widehat{U} \cong \{(x_H H) \mid x_H \in U, H \in \mathcal{P}, x_H H = x_{H'} H \text{ for } H \geq H'\} \leq \prod_{H \in \mathcal{P}} U/H.$$

Let us examine the action of U on the Tits building $\mathcal{TB}(\widehat{G})$. Let U_n be the pointwise stabiliser of the ball of radius n around the simplex \widehat{B} in $\mathcal{TB}(\widehat{G})$. Then

$$\mathcal{P}^0 := \{H \in \mathcal{P} \mid \exists n : H \geq U_n\}$$

is a basis of topology on U . We can describe the completion of U in this topology as

$$U^{crr} \cong \{(x_H H) \mid x_H \in U, H \in \mathcal{P}^0, x_H H = x_{H'} H \text{ for } H \geq H'\} \leq \prod_{H \in \mathcal{P}^0} U/H.$$

Clearly, $C(G^{crr}, U^{crr}) = 1$ because it consists of those elements $(x_H H)$ that act trivially on $\mathcal{TB}(\widehat{G})$. This forces $x_H \in U_n$ for all n and $(x_H H) = 1$. The natural map $\widehat{U} \rightarrow U^{crr}$ is the projection whose kernel is exactly $C(\widehat{G}, \widehat{U})$ that we can describe now as

$$C(\widehat{G}) = \{(x_H H) \mid x_H \in H^*, H \in \mathcal{P}, x_H H = x_{H'} H \text{ for } H \geq H'\} \leq \prod_{H \in \mathcal{P}} H^*/H$$

where $H^* := \cap_{H \subseteq K \in \mathcal{P}^0} K$. This description tells us that one of the three following statements holds:

- (1) $\mathcal{P} \setminus \mathcal{P}^0$ is finite. Then $C(\widehat{G}, \widehat{U})$ is a finite group.
- (2) $\mathcal{P} \setminus \mathcal{P}^0$ is infinite but $\{H^* \mid H \in \mathcal{P} \setminus \mathcal{P}^0\}$ is finite. Then $C(\widehat{G}, \widehat{U})$ is a finitely generated pro- p group.
- (3) $\{H^* \mid H \in \mathcal{P} \setminus \mathcal{P}^0\}$ is infinite. Then $C(\widehat{G}, \widehat{U})$ may be an infinitely generated pro- p group.

A natural question to address is whether $C(\widehat{G})$ is central. We can do it under some strong assumptions.

Lemma 3.1. *If A is irreducible of indefinite type and $q \geq n > 2$, then at least one of the following statements holds:*

- (1) $C(\widehat{G})$ is not a finitely generated pro- p group,
- (2) $C(\widehat{G}) \leq Z(\widehat{G})$.

In particular, if $C(\widehat{G})$ is finite, then $C(\widehat{G}) \leq Z(\widehat{G})$.

Proof. If A is irreducible of indefinite type and $q \geq n > 2$, then $G/Z(G)$ is a simple non-linear group as shown by Caprace and R  my [CaR].

Let us assume that $C := C(\widehat{G})$ is a finitely generated pro- p group. In this case the Frattini quotient $C/\Phi(C)$ is a finite elementary abelian p -group. Since $\Phi(C) \triangleleft \widehat{G}$, it follows immediately that $\widehat{G}/Z'(\widehat{G})$ acts on $C/\Phi(C)$. Now $\widehat{G}/Z'(\widehat{G})$ contains a dense subgroup isomorphic to $G/Z(G)$. This subgroup is simple non-linear, hence,

it must act trivially on the finite group $C/\Phi(C)$. Since the subgroup is dense, the whole $\widehat{G}/Z'(\widehat{G})$ acts trivially. Since the action is given by conjugation $g \cdot (c\Phi(C)) = (gcg^{-1})\Phi(C)$, we can say that $\widehat{G}/Z'(\widehat{G})$ centralises $C/\Phi(C)$.

Now let T be the torus of G , defined at the start of Section 2. Then $[T, C/\Phi(C)] = 1$. Let $C_i := \Phi_i(C)$, the i -th Frattini subgroup. Since C is finitely generated, C/C_i is a finite p -group, $\Phi(C/C_i) = \Phi(C)/C_i$ and $\{C_i, i \in \mathbb{N}\}$ is a fundamental system of open neighbourhoods of 1 in C [RibZ, 2.8.13]. It follows that T acts on C/C_i and centralises $(C/C_i)/\Phi(C/C_i)$. A theorem of Burnside states that a p' -automorphism of a p -group P , inducing the identity automorphism on $P/\Phi(P)$, is the identity itself [G, 5.1.4]. It follows that $[T, C/C_i] = 1$. Since $\{C_i, i \in \mathbb{N}\}$ is a fundamental system in C , it follows that $[T, C] = 1$. Obviously $[T^g, C] = 1$ for all $g \in \widehat{G}$. Therefore, $[(T^g, g \in \widehat{G}), C] = 1$. Since $Z(G)$ is a subgroup of T [CaR2, Cor. 5.14], $G = \langle T^g, g \in G \rangle \leq \langle T^g, g \in \widehat{G} \rangle$. The result now follows. \square

In some cases we can describe $C(\widehat{G})$ fully.

Proposition 3.2. *Suppose that the generalised Cartan matrix $A = (a_{ij})_{n \times n}$ is irreducible, of untwisted affine type and $n \geq 3$. Then $C(\widehat{G}) = 1$. In particular,*

$$\widehat{G} \cong G^{ma+} \cong G^{c\lambda} \cong G^{crr}$$

and $\mathcal{G}(\widehat{\mathbb{F}_q[t, t^{-1}]}) \cong \mathcal{G}(\mathbb{F}_q((t)))$.

Proof. The root datum \mathcal{D} changes only Cartan subgroup and has no effect on U or $C(\widehat{G}) = C(\widehat{G}, \widehat{U})$. Thus we may choose \mathcal{D} so that $G \cong \mathcal{G}(\mathbb{F}_q[t, t^{-1}])$ for the corresponding Chevalley group scheme \mathcal{G} . Now Lemma 7 of [CLR] gives us that \widehat{U} is a Sylow pro- p subgroup of $\mathcal{G}(\mathbb{F}_q((t)))$. This implies that $\widehat{G} \cong \mathcal{G}(\mathbb{F}_q((t)))$ which gives the desired result. \square

We expect Proposition 3.2 to hold for a twisted affine A as well. As pointed out by the referee, it would be interesting to establish whether $\widehat{G} \cong G^{crr}$ implies that G is of affine type. For instance, the isomorphism fails in rank 2 as shown in the next proposition.

Proposition 3.3. *Suppose that the generalised Cartan matrix $A = (a_{ij})_{2 \times 2}$ is not of finite type and $p > \max\{|a_{12}|, |a_{21}|\}$. Then $C(\widehat{G})$ is an infinitely-generated pro- p -group and $\{H^* \mid H \in \mathcal{P} \setminus \mathcal{P}^0\}$ is infinite.*

Proof. For such A , Morita [Mo, 3(6)] gives the description of U in $G_{\mathcal{D}}(\mathbb{F})$ for a field \mathbb{F} of characteristic 0. His description extends to the case $\mathbb{F} = \mathbb{F}_q$, as an interested reader can verify. If $\min\{|a_{12}|, |a_{21}|\} \geq 2$, then $U = U_1 * U_2$, where $U_i \cong \mathbb{F}_q[t]$ for $i = 1, 2$. If $\min\{|a_{12}|, |a_{21}|\} = 1$, then $U = U_1 * U_2$ and each U_i is a metabelian infinitely generated group.

Consider \widehat{U} . By [RibZ, 9.1.1], $\widehat{U} = \widehat{U}_1 \amalg \widehat{U}_2$, the free pro- p product of the pro- p groups \widehat{U}_1 and \widehat{U}_2 . Since each \widehat{U}_i is an infinitely-generated pro- p group, [RibZ, 9.1.15] implies that \widehat{U} is infinitely-generated.

On the other hand, as $p > \max\{|a_{12}|, |a_{21}|\}$, the results of [CR, 2.2 and 2.4] give us that U^{crr} is a finitely generated pro- p group. The proposition follows immediately. \square

Corollary 3.4. *Let $A = (a_{ij})_{n \times n}$ be an irreducible generalised Cartan matrix whose Dynkin diagram contains an infinite edge, i.e., there exists $1 \leq i \neq j \leq n$*

with $a_{ij}a_{ji} \geq 4$. Suppose that $p > \max\{|a_{ij}|, i \neq j\}$. Then $C(\widehat{G}, \widehat{U})$ is an infinitely-generated pro- p -group and $\{H^\star \mid H \in \mathcal{P}^0\}$ is infinite.

Proof. Let P be a parabolic subgroup of G whose Levi complement corresponds to the subdiagram of the Dynkin diagram Δ of G based on α_i and α_j . Then $P = U_P \rtimes L$ where

$$L = \langle X_\alpha, T \mid \alpha \in \Delta^{re} \cap \text{Span}_{\mathbb{Z}}\{\alpha_i, \alpha_j\} \rangle$$

is a Levi complement of P and $U_P = \bigcap_{g \in P} U^g$ [R, 6.2.2]. Hence, $U = U_P \rtimes U_L$ where $U_L = U \cap L$. It follows that $\widehat{U}_L \leq \widehat{U}$. Moreover, the natural isomorphism $U/U_P \xrightarrow{\cong} U_L$ yields an exact sequence

$$(4) \quad 1 \longrightarrow U_P \xrightarrow{a} U \xrightarrow{b} U_L \rightarrow 1.$$

Since pro- p -completion is a right exact functor, the sequence

$$(5) \quad \widehat{U}_P \xrightarrow{\widehat{a}} \widehat{U} \xrightarrow{\widehat{b}} \widehat{U}_L \rightarrow 1$$

is exact as well. Therefore, \widehat{U}_L is a homomorphic image of \widehat{U} . Since \widehat{U}_L is an infinitely-generated pro- p group, so is \widehat{U} .

As $p > \max\{|a_{ij}|, i \neq j\}$, the results of [CR] imply that U^{crr} is a finitely generated pro- p group. Note that $C(\widehat{G}, \widehat{U})$ is the kernel of the homomorphism $\widehat{U} \rightarrow U^{crr}$. This finishes the proof. \square

It is possible to relate the calculations of the congruence kernel of a Levi factor and of the unipotent radical of a parabolic. Let $J \subseteq I$ and $P := P_J$ a parabolic in $G_{\mathcal{D}}(\mathbb{F})$ with the unipotent radical U_P and a Levi complement $L = \langle X_\alpha, T \mid \alpha \in \Delta^{re} \cap \text{Span}_{\mathbb{Z}}(J) \rangle$. Then $P = U_P \rtimes L$ [R, 6.2.2] and we have a natural isomorphism $\widehat{L} \xrightarrow{\cong} \overline{L}^{\subseteq \widehat{G}} = \overline{L}^{\subseteq \overline{P}}$ where $\overline{P} := \overline{P}^{\subseteq \widehat{G}}$. Let $U_L := U \cap L$. Then U_L is the unipotent radical of a Borel subgroup of L . Two “parabolic” congruence kernels “approximate” $C(\widehat{G}, \widehat{U})$:

Theorem 3.5. *There exists an exact sequence of topological groups*

$$(6) \quad 1 \longrightarrow C(\widehat{G}, \overline{U_P}) \longrightarrow C(\widehat{G}, \widehat{U}) \longrightarrow C(\widehat{L}, \widehat{U}_L) \rightarrow 1.$$

Moreover, $C(\widehat{L}, \widehat{U}_L)$ is a subgroup of $C(\widehat{G}, \widehat{U})$. In particular, if $C(\widehat{L}) \neq 1$ then $C(\widehat{G}) \neq 1$.

Proof. Notice that \widehat{L} is a topological Kac-Moody group on its own letting us talk about $C(\widehat{L}) = C(\widehat{L}, \widehat{U}_L)$.

Let us examine the exact sequence (5) in the proof of Corollary 3.4. The image $\widehat{a}(\widehat{U}_P)$ is a closed subgroup containing $\widehat{a}(U_P)$. Since U_P is dense in \widehat{U}_P , $\widehat{a}(U_P)$ is dense in $\widehat{a}(\widehat{U}_P)$. This yields another exact sequence

$$(7) \quad 1 \longrightarrow \overline{U_P} \xrightarrow{\overline{a}} \widehat{U} \xrightarrow{\widehat{b}} \widehat{U}_L \rightarrow 1.$$

The same argument applied to the semidirect decomposition $P = U_P \rtimes L$ gives an exact sequence with $\widehat{c}|_{\widehat{G}} = \widehat{b}$:

$$(8) \quad 1 \longrightarrow \overline{U_P} \xrightarrow{\overline{a}} \overline{P} \xrightarrow{\widehat{c}} \widehat{L} \rightarrow 1$$

where the closure $\overline{P} = \overline{P}^{\subseteq \widehat{G}}$ is the completion of P in the uniformity induced from \widehat{G} . Loosely speaking, both \overline{P} and \widehat{G} are obtained by pro- p -completion of U . Both $\overline{U_P}$

and \widehat{U} are subgroups of \overline{P} . The map \overline{a} is the inclusion of subgroups. Conjugating them by all $g \in \overline{P}$ and then intersecting yields the inclusion $\overline{a} : C(\overline{P}, \overline{U_P}) \hookrightarrow C(\overline{P}, \widehat{U})$. Moreover, the sequence (7) restricts to a new sequence

$$(9) \quad 1 \longrightarrow C(\overline{P}, \overline{U_P}) \xrightarrow{\overline{a}} C(\overline{P}, \widehat{U}) \xrightarrow{\widehat{b}} C(\widehat{L}, \widehat{U_L}).$$

Observe that $p \cdot y := \widehat{c}(p)y\widehat{c}(p)^{-1}$, $p \in \overline{P}$, $y \in \widehat{L}$, gives a \overline{P} -action on \widehat{L} . It follows that $\widehat{b}(C(\overline{P}, \widehat{U})) \subseteq C(\widehat{L}, \widehat{U_L})$, so that the sequence (9) is well-defined.

We can conjugate $C(\overline{P}, \overline{U_P})$ and $C(\overline{P}, \widehat{U})$ by all $g \in \widehat{G}$ and intersect further. This yields a subsequence of the sequence (9):

$$(10) \quad 1 \longrightarrow C(\widehat{G}, \overline{U_P}) \xrightarrow{\overline{a}} C(\widehat{G}, \widehat{U}) \xrightarrow{\widehat{b}} C(\widehat{L}, \widehat{U_L}).$$

This sequence is precisely the sequence (6) in the statement of the theorem. It remains to establish surjectivity of \widehat{b} in the sequence (10).

Consider the restriction of the continuous homomorphism $\phi : \widehat{G} \rightarrow G^{crr}$ to \widehat{L} . Clearly, $\phi(\widehat{L}) = L^{crr}$. In particular, $\phi(C(\widehat{L}, \widehat{U_L})) \subseteq L^{crr}$. We have an \widehat{L} -equivariant map

$$\eta : \mathcal{TB}(\widehat{L})_m = \widehat{L}/(\overline{B} \cap \widehat{L}) \rightarrow \mathcal{TB}(\widehat{G})_m = \widehat{G}/\overline{B}, \quad g(\overline{B} \cap \widehat{L}) \mapsto g\overline{B},$$

where by $\mathcal{TB}(\widehat{G})_m$ and $\mathcal{TB}(\widehat{L})_m$ we denote the set of simplices of maximal dimension in the corresponding Tits buildings. As a subset of $\mathcal{TB}(\widehat{G}) = \mathcal{TB}(G^{crr})$, the image of η consists of those simplices that have P as a face because, corestricted to its image, η can be identified with the natural map $\widehat{L}/(\overline{B} \cap \widehat{L}) \rightarrow \overline{P}/\overline{B}$.

Since $C(\widehat{L}, \widehat{U_L})$ acts trivially on $\mathcal{TB}(\widehat{L})$, it follows that $C(\widehat{L}, \widehat{U_L})$ fixes the image of η . Since the stabiliser of an individual simplex is a Borel subgroup, the fixator of all these simplices is $C(\overline{P}, \overline{B})$. It follows from [CaR2, Th 6.3] that $C(\overline{P}, \overline{B})$ is equal to $U_P^{crr}T'$, where T' is a subgroup of a torus in \overline{B} . Therefore, $\phi(C(\widehat{L}, \widehat{U_L})) \subseteq U_P^{crr}T'$.

Now $C(\widehat{L}, \widehat{U_L})$ and U_P^{crr} are pro- p -groups, while T' is a finite p' -group, i.e., a group of order coprime to p . Thus, $\phi(C(\widehat{L}, \widehat{U_L})) \subseteq U_P^{crr}$. Furthermore, $\phi(C(\widehat{L}, \widehat{U_L})) \subseteq L^{crr} \cap U_P^{crr} = 1$. It follows that $C(\widehat{L}, \widehat{U_L})$ is contained in the kernel of ϕ that is equal to $Z'(\widehat{G}) = Z(G) \times C(\widehat{G}, \widehat{U})$. Since $Z(G)$ is a subgroup of the torus [CaR2, Cor. 5.14], it is a finite p' -group, while $C(\widehat{G}, \widehat{U})$ is a pro- p -group. It follows that $C(\widehat{L}, \widehat{U_L})$ is contained in $C(\widehat{G}, \widehat{U})$. This inclusion splits the sequence (10) proving surjectivity of \widehat{b} . \square

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